Correlation and regression

Notes prepared by Pamela Peterson Drake

Contents

Basic terms and concepts

1. A ***scatter plot*** is a graphical representation of the relation between two or more variables. In the scatter plot of two variables x and y, each point on the plot is an x-y pair.

2. We use regression and correlation to describe the variation in one or more variables.

   A. The ***variation*** is the sum of the squared deviations of a variable.

   $$\text{Variation}=\sum_{i=1}^{N} x-\bar{x}^{\ 2}$$

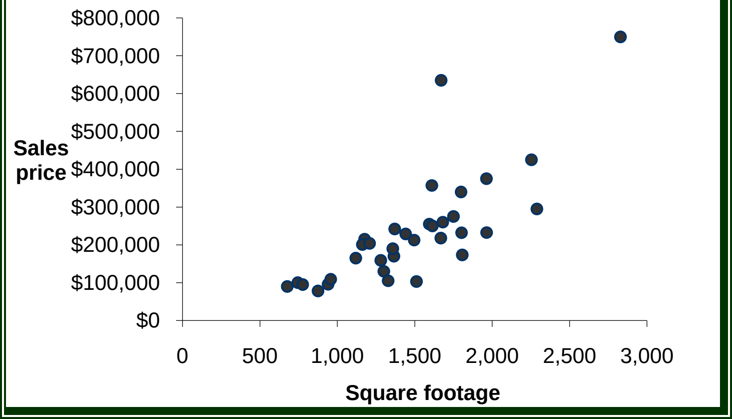
   B. The variation is the numerator of the **variance** of a sample:

   $$\text{Variance}=\frac{\sum_{i=1}^{N} x-\bar{x}^{\ 2}}{N-1}$$

   C. Both the variation and the variance are ***measures of the dispersion*** of a sample.

Example1: Home sale prices and square footage

Home sales prices (vertical axis) v. square footage for a sample of 34 home sales in September 2005 in St. Lucie County.



3. The ***covariance*** between two random variables is a statistical measure of the degree to which the two variables move together.

   A. The covariance captures how one variable is different from *its* mean as the other variable is different from *its* mean.

   B. A *positive* covariance indicates that the variables tend to move together; a *negative* covariance indicates that the variables tend to move in opposite directions.

   C. The covariance is calculated as the ratio of the ***covariation*** to the sample size less one:

   $$\text{Covariance}=\frac{\sum_{i=1}^{N}(x_i-\bar{x})(y_i-\bar{y})}{N-1}$$

   where N is the sample size
   x_i is the ith observation on variable x,
   $\bar{x}$ is the mean of the variable x observations,
   y_i is the ith observation on variable y, and
   $\bar{y}$ is the mean of the variable y observations.

   D. The actual value of the covariance is not meaningful because it is affected by the scale of the two variables. That is why we calculate the correlation coefficient – to make something interpretable from the covariance information.

   E. The ***correlation coefficient***, r, is a measure of the strength of the relationship between or among variables.

   Calculation:

Note: Correlation does <u>not</u> imply causation. We may say that two variables X and Y are correlated, but that does not mean that X causes Y or that Y causes X – they simply are related or associated with one another.

$$r = \frac{\text{covariance betwenx and y}}{\left(\begin{matrix}\text{standard deviation}\\ \text{of x}\end{matrix}\right)\left(\begin{matrix}\text{standard deviation}\\ \text{of y}\end{matrix}\right)}$$

$$r = \frac{\left(\sum_{i=1}^{N}(x_i-\bar{x})\ (y_i-\bar{y})\right) \Big/ N-1}{\sqrt{\dfrac{\sum_{i=1,n}^{N}(x_i-\bar{x})^2}{N-1}}\ \sqrt{\dfrac{\sum_{i=1}^{N}(y_i-\bar{y})^2}{N-1}}}$$

Example 2: Calculating the correlation coefficient

				Squared		Squared	
			Deviation	deviation of	Deviation	deviation of	Product of
			of x	x	of y	y	deviations
Observation	x	y	$x-\bar{x}$	$(x-\bar{x})^2$	$y-\bar{y}$	$(y-\bar{y})^2$	$(x-\bar{x})(y-\bar{y})$
1	12	50	-1.50	2.25	8.40	70.56	-12.60
2	13	54	-0.50	0.25	12.40	153.76	-6.20
3	10	48	-3.50	12.25	6.40	40.96	-22.40
4	9	47	-4.50	20.25	5.40	29.16	-24.30
5	20	70	6.50	42.25	28.40	806.56	184.60
6	7	20	-6.50	42.25	-21.60	466.56	140.40
7	4	15	-9.50	90.25	-26.60	707.56	252.70
8	22	40	8.50	72.25	-1.60	2.56	-13.60
9	15	35	1.50	2.25	-6.60	43.56	-9.90
10	23	37	9.50	90.25	-4.60	21.16	-43.70
Sum	135	416	0.00	374.50	0.00	2,342.40	445.00

Calculations:

$\bar{x}$ = 135/10 = 13.5

$\bar{y}$ = 416 / 10 = 41.6

s_x^2 = 374.5 / 9 = 41.611

s_y^2 = 2,342.4 / 9 = 260.267

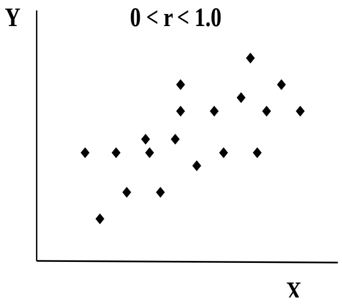
$$r = \frac{445/9}{\sqrt{41.611}\ \sqrt{260.267}} = \frac{49.444}{(6.451)(16.133)} = 0.475$$

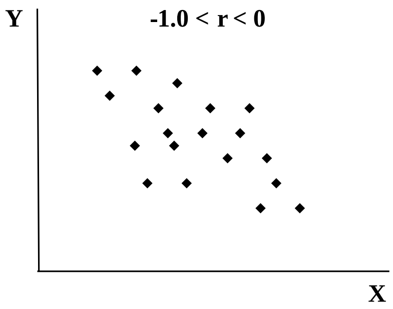
i. The type of relationship is represented by the correlation coefficient:

r =+1	perfect positive correlation
+1 >r > 0	positive relationship
r = 0	no relationship
0 > r > –1	negative relationship
r = –1	perfect negative correlation

ii. You can determine the degree of correlation by looking at the scatter graphs.

- If the relation is upward there is **positive correlation**.
- If the relation downward there is **negative correlation**.





iii. The correlation coefficient is bound by –1 and +1. The closer the coefficient to –1 or +1, the stronger is the correlation.

iv. With the exception of the extremes (that is, r = 1.0 or r = -1), we cannot really talk about the strength of a relationship indicated by the correlation coefficient without a statistical test of significance.

v. The hypotheses of interest regarding the population correlation, ρ, are:

Null hypothesis H_0: $\rho = 0$

In other words, there is no correlation between the two variables

Alternative hypothesis H_a: $\rho \neq 0$

In other words, there is a correlation between the two variables

vi. The test statistic is t-distributed with N-2 degrees of freedom:[1]

$$t = \frac{r\sqrt{N-2}}{\sqrt{1-r^2}}$$

vii. To make a decision, compare the calculated t-statistic with the critical t-statistic for the appropriate degrees of freedom and level of significance.

Example 2, continued

In the previous example,

$r = 0.475$

$N = 10$

$$t = \frac{0.475\sqrt{8}}{\sqrt{1-0.475^2}} = \frac{1.3435}{0.88} = 1.5267$$

[1] We lose two degrees of freedom because we use the mean of each of the two variables in performing this test.

Problem
Suppose the correlation coefficient is 0.2 and the number of observations is 32. What is the calculated test statistic? Is this significant correlation using a 5% level of significance?

Solution

Hypotheses:
H_0: $\rho = 0$
H_a: $\rho \neq 0$

Calculated t-statistic: $t = \frac{0.2\sqrt{32\text{-}2}}{\sqrt{1\text{-}0.04}} = \frac{0.2\sqrt{30}}{\sqrt{0.96}} = 1.11803$

Degrees of freedom = 32-2 = 30

The critical t-value for a 5% level of significance and 30 degrees of freedom is 2.042. Therefore, we conclude that there is no correlation (1.11803 falls between the two critical values of –2.042 and +2.042).

Problem
Suppose the correlation coefficient is 0.80 and the number of observations is 62. What is the calculated test statistic? Is this significant correlation using a 1% level of significance?

Solution

Hypotheses:
H_0: $\rho = 0$
H_a: $\rho \neq 0$

Calculated t-statistic: $t = \frac{0.80\sqrt{62-2}}{\sqrt{1-0.64}} = \frac{0.80\sqrt{60}}{\sqrt{0.36}} = \frac{6.19677}{0.6} = 10.32796$

The critical t-value for a 1% level of significance and 61 observations is 2.665. Therefore, we reject the null hypothesis and conclude that there is correlation.

F. An ***outlier*** is an extreme value of a variable. The outlier may be quite large or small (where large and small are defined relative to the rest of the sample).

i. An outlier may affect the sample statistics, such as a correlation coefficient. It is possible for an outlier to affect the result, for example, such that we conclude that there is a significant relation when in fact there is none or to conclude that there is no relation when in fact there is a relation.

ii. The researcher must exercise judgment (and caution) when deciding whether to include or exclude an observation.

G. ***Spurious correlation*** is the appearance of a relationship when in fact there is no relation. Outliers may result in spurious correlation.

i. The correlation coefficient does not indicate a *causal* relationship. Certain data items may be highly correlated, but not necessarily a result of a causal relationship.

ii. A good example of a spurious correlation is snowfall and stock prices in January. If we regress historical stock prices on snowfall totals in Minnesota, we would get a statistically significant relationship – especially for the month of January. Since there is not an economic reason for this relationship, this would be an example of spurious correlation.

Simple regression

1. ***Regression*** is the analysis of the relation between one variable and some other variable(s), assuming a linear relation. Also referred to as ***least squares regression*** and ***ordinary least squares*** (***OLS***).

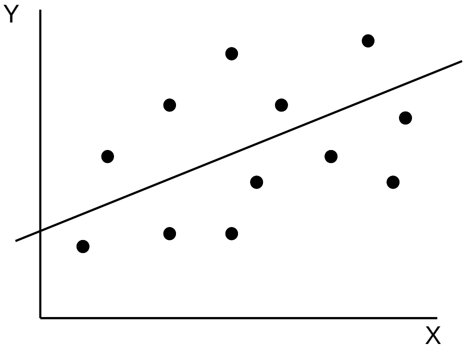
   A. The purpose is to explain the variation in a variable (that is, how a variable differs from it's mean value) using the variation in one or more *other* variables.

   B. Suppose we want to describe, explain, or predict why a variable differs from its mean. Let the ith observation on this variable be represented as Y_i, and let n indicate the number of observations.

   The variation in Y_i's (what we want to explain) is:

$$\begin{array}{c}\text{Variation}\\ \text{of Y}\end{array} = \sum_{i=1}^{N} y_i\text{-}\bar{y}^{\;2} = SS_{Total}$$

   C. The ***least squares principle*** is that the regression line is determined by minimizing the sum of the squares of the vertical distances between the actual Y values and the predicted values of Y.



   A line is fit through the XY points such that the sum of the squared residuals (that is, the sum of the squared the vertical distance between the observations and the line) is minimized.

2. The variables in a regression relation consist of dependent and independent variables.

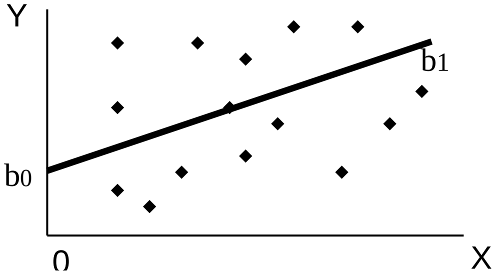
   A. The ***dependent variable*** is the variable whose variation is being explained by the other variable(s). Also referred to as the ***explained variable***, the ***endogenous variable***, or the ***predicted variable***.

   B. The ***independent variable*** is the variable whose variation is used to explain that of the dependent variable. Also referred to as the ***explanatory variable***, the ***exogenous variable***, or the ***predicting variable.***

   C. The parameters in a simple regression equation are the slope (b_1) and the intercept (b_0):

$$y_i = b_0 + b_1\, x_i + \varepsilon_i$$

where	y_i	is the ith observation on the dependent variable,
	x_i	is the ith observation on the independent variable,
	b_0	is the intercept.
	b_1	is the slope coefficient,
	ε_i	is the residual for the ith observation.



D. The ***slope***, b_1, is the change in Y for a given one-unit change in X. The slope can be positive, negative, or zero, calculated as:

$$b_1 = \frac{\text{cov}(X,Y)}{\text{var}(X)} = \frac{\sum_{i=1}^{N}(y_i - \bar{y})(x_i - \bar{x}) \Big/ N-1}{\sum_{i=1}^{N}(x_i - \bar{x})^2 \Big/ N-1}$$

Hint: Think of the regression line as the average of the relationship between the independent variable(s) and the dependent variable. The residual represents the distance an observed value of the dependent variables (i.e., Y) is away from the average relationship as depicted by the regression line.

Suppose that:

$$\sum_{i=1}^{N}(y - \bar{y})(x_i - \bar{x}) = 1{,}000$$

$$\sum_{i=1}^{N}(x_i - \bar{x})^2 = 450$$

N= 30

Then

$$\hat{b}_1 = \frac{1{,}000 \big/ 29}{450 \big/ 29} = \frac{34.48276}{15.51724} = 2.2222$$

A short-cut formula for the slope coefficient:

$$b_1 = \frac{\sum_{i=1}^{N}(y_i - \bar{y})(x_i - \bar{x}) \Big/ N-1}{\sum_{i=1}^{N}(x_i - \bar{x})^2 \Big/ N-1} = \frac{\sum_{i=1}^{N} x_i y_i - \left(\sum_{i=1}^{N} x_i \sum_{i=1}^{N} y_i\right) \Big/ N}{\left(\sum_{i=1}^{N} x_i^2 - \left(\sum_{i=1}^{N} x_i\right)^2 \Big/ N\right)}$$

Whether this is truly a short-cut or not depends on the method of performing the calculations: by hand, using Microsoft Excel, or using a calculator.

E. The ***intercept***, b_0, is the line's intersection with the Y-axis at X=0. The intercept can be positive, negative, or zero. The intercept is calculated as:

$$\hat{b}_0 = \bar{y} - b_1 \bar{x}$$

3. Linear regression assumes the following:

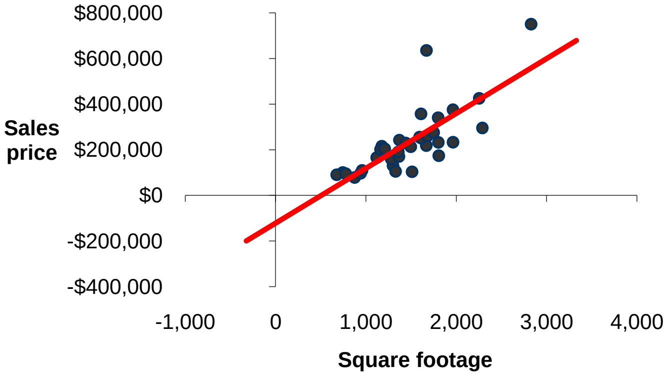
   A. A linear relationship exists between dependent and independent variable. Note: if the relation is not linear, it may be possible to transform one or both variables so that there is a linear relation.

   B. The independent variable is uncorrelated with the residuals; that is, the independent variable is not random.

   C. The expected value of the disturbance term is zero; that is, $E(\varepsilon_i)=0$

Example 1, continued:

Home sales prices (vertical axis) v. square footage for a sample of 34 home sales in September 2005 in St. Lucie County.



   D. There is a constant variance of the disturbance term; that is, the disturbance or residual terms are all drawn from a distribution with an identical variance. In other words, the disturbance terms are ***homoskedastistic***. [A violation of this is referred to as ***heteroskedasticity***.]

   E. The residuals are independently distributed; that is, the residual or disturbance for one observation is not correlated with that of another observation. [A violation of this is referred to as *autocorrelation*.]

   F. The disturbance term (a.k.a. ***residual***, a.k.a. ***error term***) is normally distributed.

4. The ***standard error of the estimate***, SEE, (also referred to as the ***standard error of the residual*** or ***standard error of the regression***, and often indicated as s_e) is the standard deviation of predicted dependent variable values about the estimated regression line.

5. Standard error of the estimate (SEE) $= \sqrt{s_e^2} = \sqrt{\dfrac{SS_{Residual}}{N-2}}$

$$SEE = \sqrt{\frac{\sum_{i=1}^{N}(y_i - \hat{b}_0 - \hat{b}_i x_i)^2}{N-2}} = \sqrt{\frac{\sum_{i=1}^{N}(y_i - \hat{y}_i)^2}{N-2}} = \sqrt{\frac{\sum_{i=1}^{N}\hat{\varepsilon}_i^{\,2}}{N-2}}$$

where $SS_{Residual}$ is the sum of squared errors;

^ indicates the predicted or estimated value of the variable or parameter; and

$\hat{y}_I = \hat{b}_0 + \hat{b}_i x_i$, is a point on the regression line corresponding to a value of the independent variable, the x_i; the expected value of y, given the estimated mean relation between x and y.

Example 2, continued

Consider the following observations on X and Y:

Observation	X	Y
1	12	50
2	13	54
3	10	48
4	9	47
5	20	70
6	7	20
7	4	15
8	22	40
9	15	35
10	23	37
Sum	135	416

The estimated regression line is:

$$y_i = 25.559 + 1.188\ x_i$$

and the residuals are calculated as:

Observation	x	y	$\hat{y}$	$y - \hat{y}$	ε^2
1	12	50	39.82	10.18	103.68
2	13	54	41.01	12.99	168.85
3	10	48	37.44	10.56	111.49
4	9	47	36.25	10.75	115.50
5	20	70	49.32	20.68	427.51
6	7	20	33.88	-13.88	192.55
7	4	15	30.31	-15.31	234.45
8	22	40	51.70	-11.70	136.89
9	15	35	43.38	-8.38	70.26
10	23	37	52.89	-15.89	252.44
Total				0	1,813.63

Therefore,

$SS_{Residaul}$ = 1813.63 / 8 = 226.70
SEE = √226.70 = 15.06

A. The standard error of the estimate helps us gauge the "fit" of the regression line; that is, how well we have described the variation in the dependent variable.

i. The smaller the standard error, the better the fit.

ii. The standard error of the estimate is a measure of close the estimated values (using the estimated regression), the $\hat{y}$'s, are to the actual values, the Y's.

iii. The ε_i's (a.k.a. the disturbance terms; a.k.a. the residuals) are the vertical distance between the observed value of Y and that predicted by the equation, the $\hat{y}$'s.

iv. The ε_i's are in the same terms (unit of measure) as the Y's (e.g., dollars, pounds, billions)

6. The ***coefficient of determination***, R^2, is the percentage of variation in the dependent variable (variation of Y_i's or the sum of squares total, SST) explained by the independent variable(s).

A. The coefficient of determination is calculated as:

$$R^2 = \frac{\text{Explained variation}}{\text{Total variation}}$$

$$= \frac{\text{Total variation} - \text{Unexplained variation}}{\text{Total variation}} = \frac{SS_{Total} - SS_{Residual}}{SS_{Total}} = \frac{SS_{Regression}}{SS_{Total}}$$

B. An R^2 of 0.49 indicates that the independent variables explain 49% of the variation in the dependent variable.

Example 2, continued

Continuing the previous regression example, we can calculate the R^2:

Observation	x	y	$(y-\bar{y})^2$	$\hat{y}$	$Y-\hat{y}$	$(\hat{y}-\bar{y})^2$	ε^2
1	12	50	70.56	39.82	10.18	3.18	103.68
2	13	54	153.76	41.01	12.99	0.35	168.85
3	10	48	40.96	37.44	10.56	17.30	111.49
4	9	47	29.16	36.25	10.75	28.59	115.50
5	20	70	806.56	49.32	20.68	59.65	427.51
6	7	20	466.56	33.88	-13.88	59.65	192.55
7	4	15	707.56	30.31	-15.31	127.43	234.45
8	22	40	2.56	51.70	-11.70	102.01	136.89
9	15	35	43.56	43.38	-8.38	3.18	70.26
10	23	37	21.16	52.89	-15.89	127.43	252.44
Total		416	2,342.40	416.00	0.00	528.77	1,813.63

R^2 = 528.77 / 2,342.40 = **22.57%**
or
R^2 = 1 – (1,813.63 / 2,342.40) = 1 – 0.7743 = **22.57%**

7. A ***confidence interval*** is the range of regression coefficient values for a given value estimate of the coefficient and a given level of probability.

A. The confidence interval for a regression coefficient $\hat{b}_1$ is calculated as:

$$\hat{b}_1 \pm t_c s_{\hat{b}_1}$$

or

$$\hat{b}_1 - t_c s_{\hat{b}_1} < b_1 < \hat{b}_1 + t_c s_{\hat{b}_1}$$

where t_c is the critical t-value for the selected confidence level. If there are 30 degrees of freedom and a 95% confidence level, t_c is 2.042 [taken from a t-table].

B. The interpretation of the confidence interval is that this is an interval that we believe will include the true parameter ($s_{\hat{b}_1}$ in the case above) with the specified level of confidence.

8. As the ***standard error of the estimate*** (the variability of the data about the regression line) rises, the confidence widens. In other words, the more variable the data, the less confident you will be when you're using the regression model to estimate the coefficient.

9. The ***standard error of the coefficient*** is the square root of the ratio of the variance of the regression to the variation in the independent variable:

$$s_{\hat{b}_1} = \sqrt{\frac{s_e^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}}$$

A. Hypothesis testing: an individual explanatory variable

i. To test the hypothesis of the slope coefficient (that is, to see whether the estimated slope is equal to a hypothesized value, b_0, Ho: $b = b_1$, we calculate a t-distributed statistic:

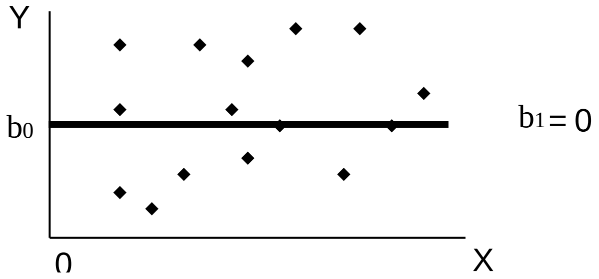
$$t_b = \frac{\hat{b}_1 - b_1}{s_{\hat{b}_1}}$$

ii. The test statistic is t–distributed with N–k–1 degrees of freedom (number of observations (N), less the number of independent variables (k), less one).

B. If the t–statistic is *greater* than the critical t–value for the appropriate degrees of freedom, (or *less* than the critical t–value for a negative slope) we can say that the slope coefficient is *different* from the hypothesized value, b_1.

C. If there is no relation between the dependent and an independent variable, the slope coefficient, b_1, would be zero.

Note: The formula for the standard error of the coefficient has the variation of the independent variable in the denominator, not the variance. The variance = variation / n-1.



- A zero slope indicates that there is no change in Y for a given change in X
- A zero slope indicates that there is no relationship between Y and X.

D. To test whether an independent variable explains the variation in the dependent variable, the hypothesis that is tested is whether the slope is zero:

Ho: $b_1 = 0$

versus the alternative (what you conclude if you reject the null, Ho):

Ha: $b_1 \neq 0$

This alternative hypothesis is referred to as a two-sided hypothesis. This means that we reject the null if the observed slope is different from zero in either direction (positive or negative).

E. There are hypotheses in economics that refer to the sign of the relation between the dependent and the independent variables. In this case, the alternative is directional (>

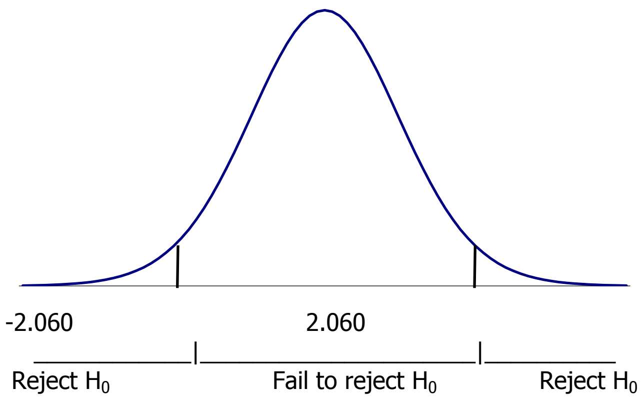
or <) and the t-test is one-sided (uses only one tail of the t-distribution). In the case of a one-sided alternative, there is only one critical t-value.

Example 3: Testing the significance of a slope coefficient

Suppose the estimated slope coefficient is 0.78, the sample size is 26, the standard error of the coefficient is 0.32, and the level of significance is 5%. Is the slope difference than zero?

The calculated test statistic is: $t_b = \dfrac{\hat{b}_1 - b_1}{s_{\hat{b}_1}} = \dfrac{0.78 - 0}{0.32} = 2.4375$

The critical t-values are ± 2.060:



Therefore, we reject the null hypothesis, concluding that the slope is different from zero.

10. Interpretation of coefficients.

    A. The estimated intercept is interpreted as the value of the dependent variable (the Y) if the independent variable (the X) takes on a value of zero.

    B. The estimated slope coefficient is interpreted as the change in the dependent variable for a given one-unit change in the independent variable.

    C. Any conclusions regarding the importance of an independent variable in explaining a dependent variable requires determining the statistical significance if the slope coefficient. Simply looking at the magnitude of the slope coefficient does not address this issue of the importance of the variable.

11. ***Forecasting*** is using regression involves making predictions about the dependent variable based on average relationships observed in the estimated regression.

    A. ***Predicted values*** are values of the dependent variable based on the estimated regression coefficients and a prediction about the values of the independent variables.

Example 4
Suppose you estimate a regression model with the following estimates:

$$\hat{y} = 1.50 + 2.5\ X_1$$

In addition, you have forecasted value for the independent variable, X_1=20. The forecasted value for y is 51.5:

$$\hat{y} = 1.50 + 2.50\ (20) = 1.50 + 50 = 51.5$$

    B. For a simple regression, the value of Y is predicted as:

$$\hat{y} = \hat{b}_0 + \hat{b}_i x_p$$

where $\hat{y}$ is the predicted value of the dependent variable, and

x_p is the predicted value of the independent variable (input).

12. An **analysis of variance table** (***ANOVA table)*** table is a summary of the explanation of the variation in the dependent variable. The basic form of the ANOVA table is as follows:

Source of variation	Degrees of freedom	Sum of squares	Mean square
Regression (explained)	1	Sum of squares regression ($SS_{Regression}$)	Mean square regression = $\frac{SS_{Regression}}{1}$
Error (unexplained)	N-2	Sum of squares residual ($SS_{Residual}$)	Mean square error = $\frac{SS_{Residual}}{N\text{-}2}$
Total	N-1	Sum of squares total (SS_{Total})	

Example 5

Source of variation	Degrees of freedom	Sum of squares	Mean square
Regression (explained)	1	5050	5050
Error (unexplained)	28	600	21.429
Total	29	5650	

$$R^2 = \frac{5,050}{5,650} = 0.8938 \text{ or } 89.38\%$$

$$\text{SEE} = \sqrt{\frac{600}{28}} = \sqrt{21.429} = 4.629$$

Multiple Regression

1. ***Multiple regression*** is regression analysis with more than one independent variable.

   A. The concept of multiple regression is identical to that of simple regression analysis except that two or more independent variables are used simultaneously to explain variations in the dependent variable.

$$y = b_0 + b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4$$

   B. In a multiple regression, the goal is to minimize the sum of the squared errors. Each slope coefficient is estimated while holding the other variables constant.

We do not represent the multiple regression graphically because it would require graphs that are in more than two dimensions.

2. The ***intercept*** in the regression equation has the same interpretation as it did under the simple linear case – the intercept is the value of the dependent variable when all independent variables are equal zero.

3. The ***slope coefficient*** is the parameter that reflects the change in the dependent variable for a <u>one unit</u> change in the independent variable.

   A. The slope coefficients (the betas) are described as the movement in the dependent variable for a one unit change in the independent variable – *holding all other independent variables constant*.

   B. For this reason, beta coefficients in a multiple linear regression are sometimes called *partial betas* or *partial regression coefficients*.

A slope by any other name …

- The slope coefficient is the *elasticity* of the dependent variable with respect to the independent variable.
- In other words, it's the *first derivative* of the dependent variable with respect to the independent variable.

4. Regression model:

$$Y_i = b_0 + b_1\,x_{1i} + b_2\,x_{2i} + \varepsilon_i$$

where:

b_j is the slope coefficient on the jth independent variable; and
x_{ji} is the ith observation on the jth variable.

   A. The degrees of freedom for the test of a slope coefficient are N-k-1, where n is the number of observations in the sample and k is the number of independent variables.

   B. In multiple regression, the independent variables may be correlated with one another, resulting in less reliable estimates. This problem is referred to as ***multicollinearity***.

5. A ***confidence interval*** for a population regression slope in a multiple regression is an interval centered on the estimated slope:

$$\hat{b}_i \pm t_c s_{\hat{b}_i}$$

or

$$\hat{b}\{ - t_c s_{\hat{b}_i} < b_i < \hat{b}_i + t_c s_{\hat{b}_i}$$

   A. This is the same interval using in simple regression for the interval of a slope coefficient.

   B. If this interval contains zero, we conclude that the slope is not statistically different from zero.

6. The assumptions of the multiple regression model are as follows:

A. A linear relationship exists between dependent and independent variables.

B. The independent variables are uncorrelated with the residuals; that is, the independent variable is not random. In addition, there is no exact linear relation between two or more independent variables. [Note: this is modified slightly from the assumptions of the simple regression model.]

C. The expected value of the disturbance term is zero; that is, $E(\varepsilon_i)=0$

D. There is a constant variance of the disturbance term; that is, the disturbance or residual terms are all drawn from a distribution with an identical variance. In other words, the disturbance terms are ***homoskedastistic***. [A violation of this is referred to as ***heteroskedasticity***.]

E. The residuals are independently distributed; that is, the residual or disturbance for one observation is not correlated with that of another observation. [A violation of this is referred to as *autocorrelation*.]

F. The disturbance term (a.k.a. residual, a.k.a. error term) is normally distributed.

G. The residual (a.k.a. disturbance term, a.k.a. error term) is what is not explained by the independent variables.

7. In a regression with two independent variables, the ***residual*** for the i^{th} observation is:

$$\varepsilon_i = Y_i - (\hat{b}_0 + \hat{b}_1 x_{1i} + \hat{b}_2 x_{2i})$$

8. The ***standard error of the estimate*** (SEE) is the standard error of the residual:

$$s_e = \text{SEE} = \frac{\sum_{t=1}^{N} \hat{\varepsilon}_t^2}{N-k-1} = \frac{\text{SSE}}{N-k-1}$$

9. The ***degrees of freedom***, df, are calculated as:

$$\text{df} = \frac{\text{number of}}{\text{observations}} - \frac{\text{number of}}{\text{independert variables}} - 1 = N-k-1 = N-(k+1)$$

A. The degrees of freedom are the number of independent pieces of information that are used to estimate the regression parameters. In calculating the regression parameters, we use the following pieces of information:

- The mean of the dependent variable.
- The mean of each of the independent variables.

B. Therefore,

- if the regression is a simple regression, we use the two degrees of freedom in estimating the regression line.
- if the regression is a multiple regression with four independent variables, we use five degrees of freedom in the estimation of the regression line.

Example 6: Using analysis of variance information

Suppose we estimate a multiple regression model that has five independent variables using a sample of 65 observations. If the sum of squared residuals is 789, what is the standard error of the estimate?

Solution

Given:

$SS_{Residual} = 789$

$N = 65$

$k = 5$

$$SEE = \frac{789}{65\text{-}5\text{-}1} = \frac{789}{59} = 13.373$$

10. ***Forecasting*** is using regression involves making predictions about the dependent variable based on average relationships observed in the estimated regression.

A. ***Predicted values*** are values of the dependent variable based on the estimated regression coefficients and a prediction about the values of the independent variables.

B. For a simple regression, the value of y is predicted as:

$$\hat{y} = \hat{b}_0 + \hat{b}_1 \hat{x}_1 + \hat{b}_2 \hat{x}_2$$

where

$\hat{y}$ is the predicted value of the dependent variable,

$\hat{b}_i$ is the estimated parameter, and

$\hat{x}_i$ is the predicted value of the independent variable

C. The better the fit of the regression (that is, the smaller is SEE), the more confident we are in our predictions.

Caution: The estimated intercept and all the estimated slopes are used in the prediction of the dependent variable value, even if a slope is not statistically significantly different from zero.

Example 7: Calculating a forecasted value

Suppose you estimate a regression model with the following estimates:

$$\hat{y} = 1.50 + 2.5\ X_1 - 0.2\ X_2 + 1.25\ X_3$$

In addition, you have forecasted values for the independent variables:

$X_1=20$ $X_2=120$ $X_3=50$

What is the forecasted value of y?

Solution

The forecasted value for Y is 90:

$$\hat{y} = 1.50 + 2.50\ (20) - 0.20\ (120) + 1.25\ (50)$$

$$= 1.50 + 50 - 24 + 62.50 = \mathbf{90}$$

11. The **F-statistic** is a measure of how well a set of independent variables, as a group, explain the variation in the dependent variable.

A. The F-statistic is calculated as:

$$F = \frac{\text{Mean squared regression}}{\text{Mean squared error}} = \frac{MSR}{MSE} = \frac{\dfrac{SS_{Regression}}{k}}{\dfrac{SS_{Residual}}{N\text{-}k\text{-}1}} = \frac{\dfrac{\sum_{i=1}^{N}(\hat{y}_i - \bar{y})^2}{k}}{\dfrac{\sum_{i=1}^{N}(y_i - \hat{y})^2}{N - k - 1}}$$

B. The F–statistic can be formulated to *test all independent variables as a group* (the most common application). For example, if there are four independent variables in the model, the hypotheses are:

H_0: $b_1 = b_2 = b_3 = b_4 = 0$

Ha: at least one $b_i \neq 0$

C. The F-statistic can be formulated to *test subsets of independent variables* (to see whether they have *incremental* explanatory power). For example if there are four independent variables in the model, a subset could be examined:

H_0: $b_1 = b_4 = 0$

Ha: b_1 or $b_4 \neq 0$

12. The ***coefficient of determination***, R^2, is the percentage of variation in the dependent variable explained by the independent variables.

$$R^2 = \frac{\text{Explained variation}}{\text{Totalvariation}} = \frac{\begin{matrix}\text{Total} \\ \text{variation}\end{matrix} - \begin{matrix}\text{Unexplained} \\ \text{variation}\end{matrix}}{\text{Total variation}}$$

$$R^2 = \frac{\sum_{i=1}^{N}(\hat{y} - \bar{y})^2}{\sum_{i=1}^{N}(y - \bar{y})^2}$$

$0 < \mathcal{R}^2 < 1$

A. By construction, R^2 ranges from 0 to 1.0

B. The ***adjusted-R²*** is an alternative to R^2:

$$R^2 = 1 - \left(\frac{N-1}{N-k}\right)(1 - R^2)$$

i. The adjusted R^2 is less than or equal to R^2 ('equal to' only when k=1).

ii. Adding independent variables to the model will increase R^2. Adding independent variables to the model may increase or decrease the adjusted-R^2 (Note: adjusted-R^2 can even be negative).

iii. The adjusted R^2 does not have the "clean" explanation of explanatory power that the R^2 has.

13. The purpose of the Analysis of Variance (ANOVA) table is to attribute the total variation of the dependent variable to the regression model (the regression source in column 1) and the residuals (the error source from column 1).

A. **SS_{Total}** is the total variation of Y about its mean or average value (a.k.a. total sum of squares) and is computed as:

$$SS_{Total} = \sum_{i=1}^{n}(y_i - \bar{y})^2$$

where $\bar{y}$ is the mean of Y.

B. **$SS_{Residual}$** (a.k.a. SSE) is the variability that is unexplained by the regression and is computed as:

$$SS_{Residual}=SSE=\sum_{i=1}^{n}(y_i-\hat{y}_i)^2=\sum\hat{\varepsilon}_i$$

where is $\hat{Y}$ the value of the dependent variable using the regression equation.

C. **SS**

C. $\mathbf{SS_{Regression}}$ (a.k.a. $SS_{Explained}$) is the variability that is explained by the regression equation and is computed as $SS_{Total} – SS_{Residual}$.

$$SS_{Regression}=\sum_{i=1}^{N}(\hat{y}_i-\bar{y})^2$$

D. MSE is the mean square error, or MSE = $SS_{Residual}$ / (N – k - 1) where k is the number of independent variables in the regression.

E. MSR is the mean square regression, MSR =$SS_{Regression}$ / k

Analysis of Variance Table (ANOVA)

Source	df (Degrees of Freedom)	SS (Sum of Squares)	Mean Square (SS/df)
Regression	k	$SS_{Regression}$	MSR
Error	N-k-1	$SS_{Residual}$	MSE
Total	N-1	SS_{Total}	

$$R^2=\frac{SS_{Regression}}{SS_{Total}}=1-\frac{SS_{Residual}}{SS_{Total}}$$

$$F=\frac{MSR}{MSE}$$

14. ***Dummy variables*** are qualitative variables that take on a value of zero or one.

A. Most independent variables represent a continuous flow of values. However, sometimes the independent variable is of a binary nature (it’s either ON or OFF).

B. These types of variables are called dummy variables and the data is assigned a value of "0" or "1". In many cases, you apply the dummy variable concept to quantify the impact of a ***qualitative variable***. A dummy variable is a dichotomous variable; that is, it takes on a value of one or zero.

C. Use one dummy variable less than the number of classes (e.g., if have three classes, use two dummy variables), otherwise you fall into the dummy variable "trap" (perfect multicollinearity – violating assumption [2]).

D. An interactive dummy variable is a dummy variable (0,1) multiplied by a variable to create a new variable. The slope on this new variable tells us the incremental slope.

15. ***Heteroskedasticity*** is the situation in which the variance of the residuals is not constant across all observations.

A. An assumption of the regression methodology is that the sample is drawn from the same population, and that the variance of residuals is constant across observations; in other words, the residuals are homoskedastic.

B. Heteroskedasticity is a problem because the estimators do not have the smallest possible variance, and therefore the standard errors of the coefficients would not be correct.

16. ***Autocorrelation*** is the situation in which the residual terms are correlated with one another. This occurs frequently in time-series analysis.

A. Autocorrelation usually appears in time series data. If last year's earnings were high, this means that this year's earnings may have a greater probability of being high than being low. This is an example of ***positive autocorrelation*.** When a good year is always followed by a bad year, this is ***negative autocorrelation***.

B. Autocorrelation is a problem because the estimators do not have the smallest possible variance and therefore the standard errors of the coefficients would not be correct.

17. ***Multicollinearity*** is the problem of high correlation between or among two or more independent variables.

A. Multicollinearity is a problem because

i. The presence of multicollinearity can cause distortions in the standard error and may lead to problems with significance testing of individual coefficients, and

ii. Estimates are sensitive to changes in the sample observations or the model specification.

B. If there is multicollinearity, we are more likely to conclude a variable is not important.

C. Multicollinearity is likely present to some degree in most economic models. ***Perfect multicollinearity*** would prohibit us from estimating the regression parameters. The issue then is really a one of degree.

18. The economic meaning of the results of a regression estimation focuses primarily on the slope coefficients.

A. The slope coefficients indicate the change in the dependent variable for a one-unit change in the independent variable. This slope can than be interpreted as an elasticity measure; that is, the change in one variable corresponding to a change in another variable.

B. It is possible to have statistical significance, yet not have economic significance (e.g., significant abnormal returns associated with an announcement, but these returns are not sufficient to cover transactions costs).

To…	**use…**
test the role of a single variable in explaining the variation in the dependent variable	the t-statistic.
test the role of all variables in explaining the variation in the dependent variable	the F-statistic.
estimate the change in the dependent variable for a one-unit change in the independent variable	the slope coefficient.
estimate the dependent variable if all of the independent variables take on a value of zero	the intercept.
estimate the percentage of the dependent variable's variation explained by the independent variables	the R^2.
forecast the value of the dependent variable given the estimated values of the independent variable(s)	the regression equation, substituting the estimated values of the independent variable(s) in the equation.

Regression terminology

Analysis of variance
ANOVA
Autocorrelation
Coefficient of determination
Confidence interval
Correlation coefficient
Covariance
Covariation
Cross-sectional
Degrees of freedom
Dependent variable
Explained variable
Explanatory variable
Forecast
F-statistic
Heteroskedasticity
Homoskedasticity
Independent variable
Intercept
Least squares regression
Mean square error
Mean square regression
Multicollinearity
Multiple regression
Negative correlation
Ordinary least squares
Perfect negative correlation
Perfect positive correlation
Positive correlation
Predicted value
R^2
Regression
Residual
Scatterplot
s_e
SEE
Simple regression
Slope
Slope coefficient
Spurious correlation
$SS_{Residual}$
$SS_{Regression}$
SS_{Total}
Standard error of the estimate
Sum of squares error
Sum of squares regression
Sum of squares total
Time-series
t-statistic
Variance
Variation

Regression formulas

Variances

$$\text{Variation} = \sum_{i=1}^{N} (x_i - \bar{x})^2 \qquad \text{Variance} = \frac{\sum_{i=1}^{N} (x_i - \bar{x})^2}{N-1} \qquad \text{Covariance} = \frac{\sum_{i=1}^{N} (x_i - \bar{x})(y_i - \bar{y})}{N-1}$$

Correlation

$$r = \frac{\left(\sum_{i=1}^{N} (x_i - \bar{x})^2 (y_i - \bar{y})^2\right) \Big/ (N-1)}{\sqrt{\frac{\sum_{i=1,n}^{N} (x_i - \bar{x})}{N-1}} \sqrt{\frac{\sum_{i=1}^{N} (y_i - \bar{y})^2}{N-1}}} \qquad t = \frac{r\sqrt{N-2}}{\sqrt{1-r^2}}$$

Regression

$y_i = b_0 + b_1 x_i + \varepsilon_i$ $\qquad$ $y = b_0 + b_1 x_1 + b_2 x_2 + b_3 x_3 + b_4 x_4 + \varepsilon_i$

$$b_1 = \frac{\text{cov}(X,Y)}{\text{var}(X)} = \frac{\sum_{i=1}^{N} (y_i - \bar{y})(x_i - \bar{x}) \Big/ N-1}{\sum_{i=1}^{N} (x_i - \bar{x})^2 \Big/ N-1} \qquad \hat{b}_0 = \bar{y} - b_1 \bar{x}$$

Tests and confidence intervals

$$s_e = \sqrt{\frac{\sum_{i=1}^{N} (y_i - \hat{b}_0 - \hat{b}_i x_i)^2}{N-2}} = \sqrt{\frac{\sum_{i=1}^{N} (y_i - \hat{y}_i)^2}{N-2}} = \sqrt{\frac{\sum_{i=1}^{N} \hat{\varepsilon}_i^{\,2}}{N-2}}$$

$$s_{\hat{b}_1} = \sqrt{\frac{s_e^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2}}$$

$$t_b = \frac{\hat{b}_1 - b_1}{s_{\hat{b}_1}} \qquad F = \frac{\text{Mean squared regression}}{\text{Mean squared error}} = \frac{\text{MSR}}{\text{MSE}} = \frac{\frac{SS_{\text{Regression}}}{k}}{\frac{SS_{\text{Residual}}}{\text{N-k-1}}} = \frac{\sum_{i=1}^{N} \frac{(\hat{y}_i - \bar{y})^2}{k}}{\sum_{i=1}^{N} \frac{(y_i - \hat{y})^2}{N-k-1}}$$

$$\hat{b}_1 - t_c s_{\hat{b}_1} < b_1 < \hat{b}_1 + t_c s_{\hat{b}_1}$$

Forecasting

$$\hat{y}=\hat{b}_0+\hat{b}_i x_{1p}+\hat{b}_i x_{2p}+\hat{b}_i x_{3p}+\ldots+\hat{b}_i x_{Kp}$$

Analysis of Variance

$$SS_{Total} = \sum_{i=1}^{n}(y_i-\bar{y})^2 \qquad SS_{Residual}=SSE=\sum_{i=1}^{n}(y_i-\hat{y}_i)^2=\sum\hat{\varepsilon}_i \qquad SS_{Regression}=\sum_{i=1}^{N}(\hat{y}_i-\bar{y})^2$$

$$R^2=\frac{SS_{Regression}}{SS_{Total}}=1-\left(\frac{SS_{Residual}}{SS_{Total}}\right)=\frac{\sum_{i=1}^{N}(\hat{y}-\bar{y})^2}{\sum_{i=1}^{N}(y-\bar{y})^2}$$

$$F=\frac{\text{Mean squared regression}}{\text{Mean squared error}}=\frac{MSR}{MSE}=\frac{\frac{SS_{Regression}}{k}}{\frac{SS_{Residual}}{N\text{-}k\text{-}1}}=\frac{\sum_{i=1}^{N}\frac{(\hat{y}_i-\bar{y})^2}{k}}{\sum_{i=1}^{N}\frac{(y_i-\hat{y})^2}{N-k-1}}$$